THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 13
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1 Lipschitz Continuity of Convex Functions

Let $X \subseteq \mathbb{R}^n$ be convex. Recall that $f : X \to \mathbb{R}$ is said to be **convex** if and only if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for any $x, y \in X$ and $\lambda \in (0,1)$.

Remarks. Let $X \subseteq \mathbb{R}^n$ be convex and $f : X \to \mathbb{R}$ be a function. We extend the definition of $f(\cdot)$ on $\mathbb{R}^n$ by

$$\bar{f}(x) := \begin{cases} f(x) & , x \in X \\ +\infty & , x \notin X \end{cases}$$

Then f is convex on X if and only if $\bar{f}$ is convex on $\mathbb{R}^n$.

We usually consider a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and its domain as:

$$\mathrm{dom}(f) := \{x \in \mathbb{R}^n : f(x) \in \mathbb{R}\}$$

Remarks. If f is convex on $\mathbb{R}^n$, then $\mathrm{dom}(f) \subseteq \mathbb{R}^n$ is convex.

Lemma 1. *Let f be convex and $x_0 \in \mathrm{int}\ \mathrm{dom} f$. Then f is locally bounded, i.e. $\exists \varepsilon > 0$ and $C > 0$ such that*

$$|f(x)| \leq C, \ \ \forall x \in B_\varepsilon(x_0) := \{x' \in \mathbb{R}^n : \|x' - x_0\|_2 \leq \varepsilon\}.$$

Remarks. Recall the ℓ_2-norm and ℓ_∞ sup-norm as follows: $\|x\|_2 := \sqrt{\sum_{i=1}^n x_i^2}$ and $\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$.

Proof. Let us define the hypercube as

$$H_\varepsilon(x_0) := \{x' \in \mathbb{R}^n : \|x' - x_0\|_\infty \leq \varepsilon\}$$

Then, we can see $B_\varepsilon(x_0) \subseteq H_\varepsilon(x_0)$. This implies that

$$\begin{aligned} \sup_{x \in B_\varepsilon(x_0)} |f(x)| &\leq \sup_{x \in H_\varepsilon(x_0)} |f(x)| \\ &\leq \max_{e \in E} |f(x_0 \pm \varepsilon e)| := C \end{aligned}$$

where $e = (\pm 1, \pm 1, \cdots, \pm 1)$ and $C < +\infty$ for $\varepsilon > 0$ is small enough since $H_\varepsilon(x_0) \subseteq \mathrm{int}\ \mathrm{dom}(f)$. □

Lemma 2. *Let f be convex and $x_0 \in \mathrm{int}\ \mathrm{dom}(f)$. Then there exists $\varepsilon > 0$, $\tilde{C} > 0$ such that*

$$\frac{|f(x) - f(y)|}{\|x - y\|} \leq \tilde{C}$$

for all $x, y \in B_\varepsilon(x_0)$ and $x \neq y$.

Proof. Since $x_0 \in \operatorname{int} \operatorname{dom}(f)$, then by the previous lemma, there exists $\varepsilon > 0$ and $C > 0$ such that $B_{2\varepsilon}(x_0) \subseteq \operatorname{int} \operatorname{dom}(f)$ and $|f(x)| \leq C$ for all $x \in B_{2\varepsilon}(x_0)$.
Next, let z be the intersection of the line $x \to y$ and $\partial B_{2\varepsilon}(x_0)$.
Then, it is clear that $\|x - y\|_2 \leq 2\varepsilon$, $\|y - z\| \geq \varepsilon$, $\|z - x\| \geq \varepsilon$ and

$$y = \alpha z + (1 - \alpha)x$$

with $\alpha := \dfrac{\|y - x\|}{\|z - x\|} \in (0, 1)$. By the convexity of f, we have

$$\begin{aligned}
& f(y) \leq \alpha f(z) + (1 - \alpha) f(x) \\
\implies & f(y) - f(z) \leq \alpha \left(f(z) - f(x)\right) \\
& \qquad = \frac{\|y - x\|}{\|z - x\|} \left(f(z) - f(x)\right) \\
\implies & \frac{f(y) - f(z)}{\|y - x\|} \leq \frac{f(z) - f(x)}{\|z - x\|} \leq \frac{2C}{\varepsilon}
\end{aligned}$$

By symmetry, we also have

$$f(x) - f(y) \leq \frac{2C}{\varepsilon} \|x - y\| \implies \frac{|f(x) - f(y)}{\|x - y\|} \leq \frac{2C}{\varepsilon} := \tilde{C}$$

and thus completes the proof. □

Theorem 3. *Let f be convex and $K \subseteq \operatorname{int} \operatorname{dom}(f)$ be closed and bounded set. Then f is uniformly bounded on K, and f is Lipschitz on K.*

Proof. Since K is compact, and $\{B_{\varepsilon(x_0)}(x_0) : x_0 \in K\}$ is an open cover of K, then there exist a finite subcover $B_{\varepsilon_i}(x_i),\ i = 1, \ldots, m$ by the compactness of K.
Since $|f(x)| \leq C_i$ for all $x \in B_{\varepsilon_i}(x_i)$, so this implies that $|f(x)| \leq \max_{i=1,\ldots,m} C_i$, this shows that f is uniformly bounded on K.
Similarly, we can prove the Lipschitz property of f on K by applying the previous lemma. □

2 Example

Example 1. All three assumptions on K, i.e. closedness, boundedness and $K \subset \operatorname{int} \operatorname{dom}(f)$ are essential, we can see from the following examples:

1. Consider the function

$$f(x) = \begin{cases} x^2 & , \ x \in (-1, 1) \\ 2 & , \ x = \pm 1 \\ +\infty & , \ x \notin [-1, 1] \end{cases}$$

   with $\operatorname{dom}(f) = [-1, 1]$ and it is a convex function. Then f is Lipschitz on $(-1, 1) = \operatorname{int} \operatorname{dom}(f)$. However, f is not Lipschitz on $[-1, 1]$.

2. Consider the function

$$f(x) = \begin{cases} \dfrac{1}{x} & , \ x > 0 \\ +\infty & , \ x \leq 0 \end{cases}$$

   f is convex function since $f''(x) = 2x^{-3} > 0$ for any $x \in \operatorname{dom}(f) = (0, +\infty)$.
   f is not uniformly bounded on $\operatorname{int} \operatorname{dom}(f)$ and not Lipschitz on $\operatorname{int} \operatorname{dom}(f)$.

3. Consider the function $f(x) = x^2$ and $\mathrm{dom}(f) = \mathbb{R}$.
   f is not uniformly bounded on $\mathbb{R}$ and not Lipschitz on $\mathbb{R}$ since $\mathbb{R}$ is not bounded.

4. Consider the function
$$f(x) = \begin{cases} -\sqrt{x} & , \ x \geq 0 \\ +\infty & , \ x < 0 \end{cases}$$
   with $\mathrm{dom}(f) = [0, +\infty)$ and $\mathrm{int}\ \mathrm{dom}(f) = (0, +\infty)$.
   $K = [0, 1]$ is bounded and closed, but f is not Lipschitz on K since $K \not\subset \mathrm{int}\ \mathrm{dom}(f)$.

Recall: Subgradient

Definition 1. Let X be a convex set and $f : X \to \mathbb{R}$ be a function. A vector $w \in \mathbb{R}^n$ is called a **subgradient** of f at point $x \in X$ if
$$f(y) \geq f(x) + w^T(y - x), \ \ \forall y \in X$$
We denote $\partial f(x) = \{\text{all subgradient of } f \text{ at } x\}$.

Let f be a convex function and $\partial f(x) \neq \emptyset$ for all $x \in \mathrm{ri}\ \mathrm{dom}(f)$.

Lemma 4. *The following statements hold for a convex function f.*

1. *Let $x_0 \in \mathrm{int}\ \mathrm{dom}(f)$ so that f is locally Lipschitz in the sense that $\exists \varepsilon > 0, \tilde{C} > 0$ such that*
$$|f(x) - f(y)| \leq \tilde{C}\|x - y\|, \ \ \forall x, y \in B_\varepsilon(x_0)$$
   Then $\|v\| \leq \tilde{C}$ for all $v \in \partial f(x), \ \forall x \in B_\varepsilon(x_0)$.

2. *Let $v \in \partial f(x)$ and $x \in \mathrm{dom}(f)$ such that $\|v\| \leq C$, then*
$$f(x) - f(y) = \langle v, x - y \rangle \leq C\|x - y\|$$
   for any $y \in \mathrm{dom}(f)$.

Proof. 1. For $\delta > 0$ small enough, we put $y = x + \delta v' \in B_\varepsilon(x_0)$, then we compute
$$\delta\|v\|^2 = \langle v, y - x \rangle \leq f(y) - f(x) \leq \tilde{C}\|y - x\| = \tilde{C}\delta\|v\|$$
   Dividing both sides by $\delta\|v\|$ and get $\|v\| \leq \tilde{C}$ for all $v \in \partial f(x), \ x \in B_\varepsilon(x_0)$.

2. Since $v \in \partial f(x)$, by definition, we have
$$f(y) - f(x) \geq \langle v, y - x \rangle, \ \ \forall y \in \mathrm{dom}(f)$$
   Multiplying -1 on both sides yields
$$f(x) - f(y) \leq \langle v, x - y \rangle \leq \|v\|\|x - y\| \leq C\|x - y\|$$
   for any $y \in \mathrm{dom}(f)$.

□

Remarks. Let f be convex and differentiable at $x \in \mathrm{int}\ \mathrm{dom}(f)$. Then
$$f\left((1 - \lambda)x + \lambda y\right) \leq \lambda f(y) + (1 - \lambda)f(x).$$
Simply rewrite the above as
$$\begin{aligned} f(y) &\geq \frac{1}{\lambda}\left[f\left((1 - \lambda)x + \lambda y\right) - (1 - \lambda)f(x)\right] \\ &= f(x) + \frac{1}{\lambda}\left[f\left((1 - \lambda)x + \lambda y\right) - f(x)\right] \end{aligned}$$
Taking limit $\lambda \to 0$, we have $\nabla f(x) \in \partial f(x)$.

Below, we provide some basic subgradient calculus for convex functions. Observe that many of them mimic the calculus for gradient computation.

1. Scaling: $\partial(af) = a\partial f$ provided that $a > 0$.
   The condition $a > 0$ makes function f remain convex.

2. Addition: $\partial(f_1 + f_2) = \partial(f_1) + \partial(f_2)$, where f_1, f_2 are convex functions.

Proof. 1. Trivial.

2. We only show one direction proof.
   Let $v_1 \in \partial f_1(x)$ and $v_2 \in \partial f_2(x)$, then

$$\begin{cases} f_1(y) \geq f_1(x) + \langle v_1, y - x \rangle \\ f_2(y) \geq f_2(x) + \langle v_2, y - x \rangle \end{cases}$$

   Summing these two inequalities implies

$$(f_1 + f_2)(y) \geq (f_1 + f_2)(x) + \langle v_1 + v_2, y - x \rangle$$

   Therefore, we have $\partial(f_1) + \partial(f_2) \subseteq \partial(f_1 + f_2)$.

$\square$

— End of Lecture 13 —